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Stress intensity factors of a crack in a composed circular cylinder

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Abstract

A composed circular cylinder, formed by a core circular cylinder, containing a crack and enclosed by a layered hollow circular cylinder, is investigated in regard to the evaluation of stress intensity factors. Analytic solutions to the problem are provided, with which the upper and lower bounds of stress intensity factors in a cracked circular cylinder, the stress distribution in a layered hollow circular cylinder, and the stress intensity factors for a crack in the composed circular cylinder can precisely be determined. Numerical materials, demonstrating the discrete values of the stress intensity factors, as well as the general pattern according to which the stress intensity factors vary with the material and geometric constants, are presented. The solutions are developed based on a simplified and modified solution to the Hilbert problem, and the matrix presentation and manipulation of functions and variables, used in the circuit theory.

1. Introduction

The present article mainly concerns with a crack embedded in a circular cylinder, which is enclosed by a number of hollow circular cylinders of different thicknesses and materials. Due to the rising application of composite materials, this problem is of practical significance in composite material engineering as well as of theoretical interest in heterogeneous elasticity.

A primitive and special form of our problem, is a radial crack contained in a circular cylinder with a free or traction circumferential boundary. The problem has been considered by Bowie and Neal (1970), Rooke and Tweed (1972), Delale and Erdogan (1982), Shangchow (1983, 1988) and others, by the use of the methods of modified collocation, integral transform, dislocation modelling and alternating procedure. Recently, a more complicated problem, that is, a cracked circular

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cylinder enclosed by another circular cylinder of different material, has been discussed by Xu (1993). On the other hand, despite that a fairly large amount of publications exist on this subject, investigations dealing with cracked circular cylinders seem having been largely restricted to the case of a single cylinder with a simple (prescribed traction or free) circumferential boundary.

This article extends the investigation on this subject to a fully heterogeneous case, where the cracked circular cylinder is enclosed by an arbitrary number of circular cylinders of different geometries and materials. This model is of direct interest in the fracture analysis of components of composite materials; moreover, by a suitable selection of the geometry and material assortment of the enclosing circular cylinders, it can be used to simulate a single cracked circular cylinder with a circumferential boundary, which is elastically constrained by a support with an arbitrary flexibility.

To solve the problem, first a solution is developed for an elastic medium having a crack, on whose surfaces normal traction is prescribed. This solution contains a sufficiently large number of undetermined coefficients, which can be adjusted to cater for any types of boundary geometries and boundary conditions of the elastic cracked medium, provided there is no stress singularity in the medium except at the crack tips. As a very preliminary illustration, this solution is used to solve the problem of a cracked circular cylinder with a free or fixed circumferential boundary.

On the other side of the development, the boundary value problem of a layered hollow circular cylinder, consisting of an arbitrary number of sub-circular cylinders of different geometries and materials, is considered, and an exact and succinct solution in matrix form is worked out. This solution is effective, since it virtually reduces the considered heterogeneous problem into a homogeneous one of a single hollow circular cylinder.

The cracked circular cylinder and the layered hollow circular cylinder are now assembled by perfect bonding to form a composed circular cylinder with a crack. The two solutions mentioned above are accordingly combined and matched, to yield a solution to the problem of the crack in the composed circular cylinder. This solution is rigorous and, as manifested in a number of numerical examples, amenable to numerical manipulation.

The solutions are developed by using a modified and simplified solution to the Hilbert problem (Muskhelishvili, 1953, 1956), and the matrix presentation and manipulation of functions and variables, originally employed in the electric circuit theory (Carslaw and Jaeger, 1959).

2. Crack problem and solution

An infinite medium containing a crack of length b–a is shown in Fig. 1. The medium is stressed in a plane strain or plane stress state by boundary tractions, imposed on the crack surfaces and expressed by

$$\sigma_y = f(t), \quad \tau_{xy} = 0, \tag{1}$$

where t denotes the x coordinate on the crack, $a \leqslant t \leqslant b$, and $f(t)$ is an arbitrarily given, real continuous function.

It is known that the stresses in the problem can be determined from the two analytic functions, $\Phi(z)$ and $\Omega(z)$, defined in the split medium and satisfying the following equations (Muskhelishvili, 1953)

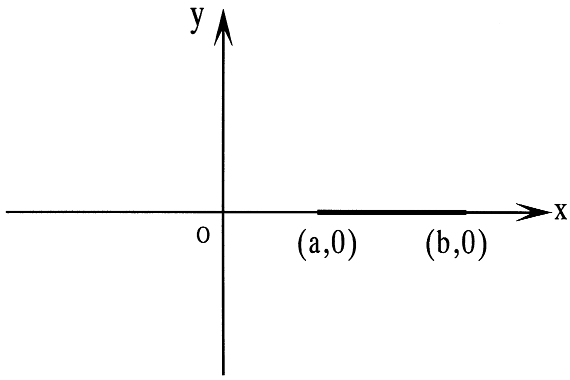

Fig. 1. The infinite medium and the crack.

$$[\Phi(t)+\Omega(t)]^{+}+[\Phi(t)+\Omega(t)]^{-}=2f(t), \tag{2}$$
$$[\Phi(t)-\Omega(t)]^{+}-[\Phi(t)-\Omega(t)]^{-}=0,$$

on the crack surfaces. One can easily see that $\Phi^*(z)$ and $\Omega^*(x)$, the particular solutions to eqns (2), are simply

$$\Phi^*(z)=\Omega^*(z)=f(z)/2. \tag{3}$$

The nonvanishing, homogeneous solution of eqns (2), denoted as $\Phi_h(z)$ and $\Omega_h(z)$, is now sought. Since, as can be seen from the second of eqns (2), $\Phi(z)-\Omega(z)$ is a function holomorphic over the whole plane, it must be a polynomial (Macrobert, 1955)

$$\Phi(z)-\Omega(z)=a_0+a_1z+a_2z^2+\cdots+a_mz^m. \tag{4}$$

Substituting eqns (3, 4) into the first of eqns (2) yields

$$\Phi_h^{+}(t)+\Phi_h^{-}(t)=a_0+a_1t+a_2t^2+\cdots+a_mt^m. \tag{5}$$

The particular and nonvanishing homogeneous solutions to the above eqn (5) are, respectively,

$$\Phi_h^*(z)=(a_0+a_1z+a_2z^2+\cdots+a_mz^m)/2, \tag{6}$$

and

$$\Phi_{hh}(z)=(c_0+c_1z+c_2z^2+\cdots+c_nz^n)/X(z), \tag{7}$$

where $c_0, c_1, \ldots, c_n$ are constants and $X(z)$ stands for a single-valued branch of function $\sqrt{(z-a)(z-b)}$, specified by

$$[X(z)]^{-1}=(B_0+B_1/z+B_2/z^2+\cdots)/z, \tag{8}$$

where B_0, B_1, ... are real constants given in the Appendix.

Equations (6, 7) complete the solution of eqn (5). Consequently, the final form of the solution to the problem, turns out to be

$$\Phi(z) = \frac{f(z)}{2} + \frac{a_0 + a_1 z + \cdots + a_m z^m}{2} + \frac{c_0 + c_1 z + \cdots + c_n z^n}{X(z)},$$

$$\Phi(z) = \frac{f(z)}{2} - \frac{a_0 + a_1 z + \cdots + a_m z^m}{2} + \frac{c_0 + c_1 z + \cdots + c_n z^n}{X(z)}. \tag{9}$$

To guarantee that the displacement in the problem is single-valued, or equivalently, the displacement increment around a closed contour enclosing the crack vanishes, by dint of the following formula (Muskhelishvili, 1953)

$$2\mu(u+iv) = \kappa \int \Phi(z)\,\mathrm{d}z - \int \Omega(\bar{z})\,\mathrm{d}\bar{z} - (z-\bar{z})\overline{\Phi(z)}, \tag{10}$$

the following equation

$$c_0 B_0 + c_1 B_1 + \cdots + c_n B_n = 0 \tag{11}$$

must hold as a supplement to eqn (9). In eqn (10), $\kappa = 3-4v$ for plane strain, $(3-v)/(1+v)$ for plane stress, μ stands for the shear modulus and v is the Poisson's ratio of the material.

Since the geometry and the external loading in the problem are symmetric to the x-axis, a simple analysis discloses that all the constant coefficients $a_0, a_1, \ldots, a_m$ and $c_0, c_1, \ldots, c_n$ are real. Moreover, when the y-axis is also an axis of symmetry, there must be

$$a_1 = a_3 = a_5 = \cdots = 0, \quad c_0 = c_2 = c_4 = \cdots = 0. \tag{12}$$

The solution (9), (11) is developed by following the basic line of Muskhelishvili, who, by the use of the integral of Cauchy's type, furnished a solution to eqns (2) in relatively complicated formulae (Muskhelishvili, 1953, 1956). The solution derived herein is simple in form, and has been obtained elementarily and directly. Besides, the polynomial with coefficients a_i, included in eqns (9), considerably extends the capability of the solution in solving crack problems in plane elasticity.

In previous studies, wherein solutions to eqns (2) are concerned, terms with coefficients c_i [see eqns (9)] are so determined as to guarantee stress regularity at infinity and to satisfy eqn (11). In fact, these terms are useful and of particular interest in solving crack problems with elastically constrained boundaries, as can be seen from the following two special cases.

Consider the case when a crack, shown in Fig. 2 and opened by the normal traction $\sigma_y = f(t)$ [see eqn (1)], is embedded radially and centrally in a circular cylinder of unit radius and with a free boundary. To this problem the general solution (9), (11) applies, but the coefficients a_i and c_i contained therein should be determined via the free boundary condition on the circumferential boundary. To do this, we suppose the y-axis is also a symmetric axis to the problem, and expand $f(t)$ into a polynomial of t such as

$$f(t) = \sum_{n=0,2,4,\ldots}^{2S} f_n t^n, \tag{13}$$

all other cases can be treated similarly. Substitution of eqns (9), (11), (13) into the following formulae (Muskhelishvili, 1953)

$$\sigma_r - i\tau_{r\theta} = \Phi(z) + \overline{\Phi(z)} - \mathrm{e}^{2i\theta}[(\bar{z}-z)\Phi'(z) - \Phi(z) + \bar{\Phi}(z)], \quad \sigma_r - i\tau_{r\theta}|_{|z|=1} = 0 \tag{14}$$

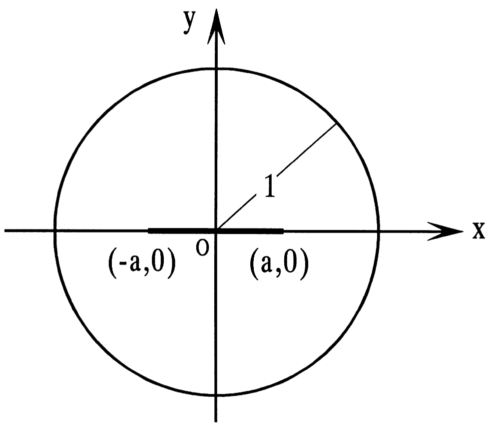

Fig. 2. The circular cylinder and the crack.

results in an equation, whose one side vanishes, while the other side stands for the sum of various terms of $A_q e^{iq\theta}$, where $q = 0, \pm 2, \pm 4, \ldots$. To satisfy the equation, A_q, the coefficient of each term, must be zero. This gives:

$$a_0 + 2\sum_{k=0}^{n}(B_{2k} - B_{2k+2})c_{2k+1} = -f_0,$$

$$pa_{2p-2} + \frac{1-2p}{2}a_{2p} + \sum_{k=0}^{n}[(1-2p)B_{2k-2p} + 2(p-1)B_{2k-2p+2} + B_{2k+2p}]c_{2k+1}$$

$$= \frac{2p-1}{2}f_{2p} - pf_{2p-2},$$

$$\frac{a_{2p}}{2} + \sum_{k=0}^{n}[B_{2k-2p} + (2p+1)B_{2k+2p} - 2(p+1)B_{2k+2p+2}]c_{2k+1} = -\frac{f_{2p}}{2}, \tag{15}$$

where $p \geqslant S$, $p = 1, 2, \ldots, n+1$, $B_j = 0$ for $j < 0$ and $f_{2p} = 0$ for $p > S$.

The unknown coefficients $a_0, a_2, \ldots, a_{2n+2}$ and $c_1, c_3, \ldots, c_{2n+1}$ can now be determined from eqns (15). In this, as well as in the following case, the single-valuedness condition of displacement, eqn (11), is automatically satisfied.

This is a problem involving multiple boundaries. Previously it was solved essentially by the superposition of two different solutions: (a) one for a crack embedded in an infinite medium, and (b) the other for a solid circular cylinder without any crack. The solution developed herein seems an exceptional one, in the sense that it consists of merely one solution, the solution (a).

Next consider the problem when the outer boundary is fixed. In this case, using eqn (10) for $|z| = 1$, $u + iv = 0$ and following a procedure analogous to that for the free boundary case, the following equations are established, to fix the unknown constants $a_0, a_2, \ldots, a_{2n}$ and $c_1, c_3, \ldots, c_{2n+1}$:

$$\frac{1}{2}[\kappa-\delta(p-1)]a_{2p-2}+\sum_{k=0}^{n}[\kappa B_{2k-2p+2}+2pB_{2p+2k}+(1-2p)B_{2p+2k-2}]c_{2k+1}$$
$$=\frac{1}{2}[\delta(p-1)-\kappa]f_{2p-2},$$
$$\frac{2p-1}{2}a_{2p}-pa_{2p-2}+\sum_{k=0}^{n}[\kappa B_{2p+2k}+(2p-1)B_{2k-2p}-2(p-1)B_{2k-2p+2}]c_{2k+1}$$
$$=\left(\frac{1}{2}-p\right)f_{2p}+(p-1)f_{2p-2}, \tag{16}$$

where $p = 1, 2, \ldots, n+1$. In eqn (16), $\delta(p-1)$ stands for a function defined as

$$\delta(p-1)=\begin{cases}1, & p=1;\\ 0, & p\neq 1.\end{cases} \tag{17}$$

The material presented in this section is based on the Muskhelishvili method. But we derive the solution (9) through simpler steps and present it in a simpler form. In addition, the solution (13)–(17) for a circular cylinder containing a crack has been obtained by a method, uniquely developed in this article.

3. Layered circular cylinder problem and solution

Figure 3 shows a layered hollow circular cylinder, consisting of N layers of different materials. Its traction boundary value problem is now investigated.

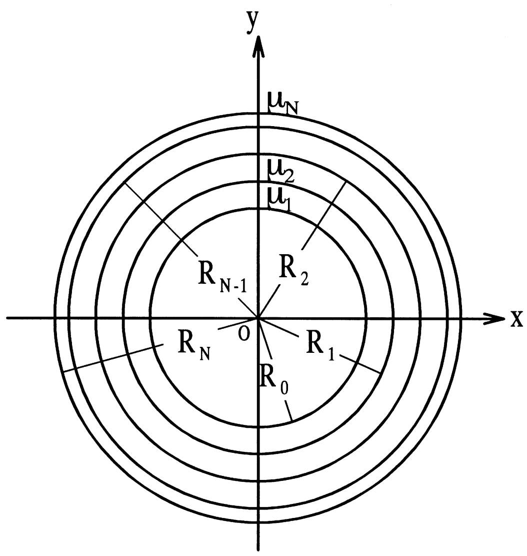

Fig. 3. The layered hollow circular cylinder.

Since the solution for a homogeneous hollow circular cylinder is known (Muskhelishvili, 1953), that for the layered cylinder can be obtained, exactly and straightforwardly, by directly combining the homogeneous solutions, and using continuity conditions of displacement and traction at the interfaces of different layers. However, this approach may lead to tedious computations and cumbersome formulas. A succinct and exact form of the solution is therefore desirable. For which we first consider the following two subsidiary problems.

Subsidiary Problem A. Consider the innermost layer cylinder individually and separately. Suppose on its inner surface, where $r = R_0$, tractions as well as displacements are expressed by

$$\sigma_r - i\tau_{r\theta} = \sum_{-\infty}^{\infty} A_k \, \mathrm{e}^{ik\theta}, \quad 2\mu_1(u+iv) = \sum_{-\infty}^{\infty} B_k \, \mathrm{e}^{ik\theta}, \tag{18}$$

under the condition that both of them are symmetric to the x-axis. As a result, all the coefficients A_k and B_k in eqns (18) must be real.

It is known (Muskhelishvili, 1953) that the stresses and displacements in this layer cylinder can be determined from two stress functions, $\Phi(z)$ and $\Psi(z)$,

$$\Phi(z) = \sum_{-\infty}^{\infty} \alpha_k z^k, \quad \Psi(z) = \sum_{-\infty}^{\infty} \beta_k z^k, \tag{19}$$

where α_k and β_k are real, through the following two formulae

$$\begin{aligned} \sigma_r - i\tau_{r\theta} &= \Phi(z) + \overline{\Phi(z)} - \mathrm{e}^{2i\theta}[\bar{z}\Phi'(z) + \Psi(z)], \\ 2\mu_1(u+iv) &= \kappa_1 \int \Phi(z)\,\mathrm{d}z - z\overline{\Phi'(z)} - \int \overline{\Psi(z)}\,\mathrm{d}\bar{z}. \end{aligned} \tag{20}$$

Substituting eqns (19) into (20) and using eqns (18), expressions for α_k's and β_k's are obtained as follows:

$$\begin{aligned} \alpha_{-1} &= \frac{A_1 R_0}{1+\kappa_1}, \quad \alpha_0 = \frac{1}{1+\kappa_1}\left(A_0 + \frac{B_1}{R_0}\right), \\ \alpha_1 &= \frac{2\kappa_1 A_1}{(1+\kappa_1)R_0}\log R_0 - \frac{B_0}{R_0^2}, \\ \beta_{-1} &= -\frac{\kappa_1 R_0 A_1}{1+\kappa_1}, \quad \beta_{-2} = \frac{2R_0 B_1}{1+\kappa_1} + \frac{1-\kappa_1}{1+\kappa_1} R_0^2 A_0, \\ \beta_{-3} &= \frac{2(1+\kappa_1 \log R_0)R_0^3 A_1}{(1+\kappa_1)} - B_0 R_0^2 - A_{-1} R_0^3, \end{aligned} \tag{21}$$

for $n = 2, 3, 4, \ldots,$

$$\alpha_n = \frac{R_0 A_{-n} + (1+n)B_{n+1}}{(1+\kappa_1)R_0^{n+1}}, \quad \alpha_{-n} = \frac{[R_0 A_n + (1-n)B_{1-n}]R_0^{n-1}}{(1+\kappa_1)},$$

$$\beta_{n-2} = \frac{(1-n)R_0A_{-n} - \kappa_1 R_0 A_n + (1-n^2)B_{1+n} + (1-n)B_{1-n}}{(1+\kappa_1)R_0^{n-1}},$$

$$\beta_{-n-2} = \frac{R_0^{n+1}[(1+n)R_0A_n - \kappa_1 R_0 A_{-n} + (1-n^2)B_{1-n} + (1+n)B_{1+n}]}{1+\kappa_1}. \tag{22}$$

All the above coefficients are determined by comparing and equating similar terms of $e^{iq\theta}$ ($q = 0, \pm 1, \pm 2, \ldots$) in eqns (20), when eqns (19) are substituted into eqns (20) and a reference is made of eqns (18). However, in fixing α_{-1} and β_{-1}, the single-valuedness condition of displacement, in the form of

$$\kappa_1 \alpha_{-1} + \beta_{-1} = 0, \tag{23}$$

has been used. Besides, it is found that the coefficient β_{-3} must satisfy the following two equations

$$\beta_{-3} = 2R_0^2 a_{-1} + R_0^4 a_1 - R_0^3 A_{-1}, \quad \beta_{-3} = 2R_0^2 a_{-1} - \kappa_1 R_0^4 a_1 + 2R_0^2 B_2. \tag{24}$$

To identify the above two equations and therefore render β_{-3} in a unified form as shown in eqns (21), the following relation

$$(2\kappa_1 R_0 \log R_0)A_1 - (1+\kappa_1)B_0 = R_0 A_{-1} + 2B_2 \tag{25}$$

must hold, as a constraint on the coefficients A_k and B_k in eqns (18).

The first one of eqns (18) explicitly prescribes the external tractions applied on $r = R_0$. The second one, while specifying the displacements on the same boundary, implicitly prescribes the external tractions on $r = R_1$, the other boundary. The external tractions applied on the two boundaries must form an equilibrium system. This requirement necessitates the relation (25).

If the external load is also symmetric to the y-axis, then it is easy to see that

$$A_{\pm 1} = A_{\pm 3} = A_{\pm 5} = \cdots = 0, \quad B_0 = B_{\pm 2} = B_{\pm 4} = \cdots = 0,$$

$$\alpha_{\pm 1} = \alpha_{\pm 3} = \alpha_{\pm 5} = \cdots = 0, \quad \beta_{\pm 1} = \beta_{\pm 3} = \beta_{\pm 5} = \cdots = 0. \tag{26}$$

Subsidiary Problem B. In this subsidiary problem we treat two adjoining layer cylinders numbered j and $j+1$ in the layered hollow cylinder. The coefficients $\alpha_k^{(j+1)}$ and $\beta_k^{(j+1)}$ in the two basic stress functions

$$\Phi_{j+1}(z) = \sum_{-\infty}^{\infty} \alpha_k^{(j+1)} z^k, \quad \Psi_{j+1}(z) = \sum_{-\infty}^{\infty} \beta_k^{(j+1)} z^k, \tag{27}$$

for the $(j+1)$th layer cylinder are determined, providing that the corresponding coefficients $\alpha_k^{(j)}$ and $\beta_k^{(j)}$ for the ith layer cylinder are known. The stresses and displacements in both the layer cylinders are supposed to be symmetric with respect to the x-axis, and $\kappa_j \alpha_{-1}^{(j)} + \beta_{-1}^{(j)} = 0$ holds to guarantee a single-valued displacement field within the jth layer cylinder.

Since the bonding between the two layer cylinders is perfect, continuity in stresses and displacements occurs at the interface $r = R_j$. Substituting Φ_j, Ψ_j and Φ_{j+1}, Ψ_{j+1}, respectively, into the first one of eqns (20) for $|z| = R_j$ results in a pair of equations. Another pair of equations are obtained, when these stress functions are substituted into the second one of eqns (20) for $|z| = R_j$.

Comparing and equating similar terms of $e^{iq\theta}$ ($q = 0, \pm 1, \pm 2, \ldots$) respectively, in the two pairs of equations, expressions for $\alpha_k^{(j+1)}$ and $\beta_k^{(j+1)}$ are obtained as follows:

$$\begin{bmatrix} \alpha_0^{(j+1)} \\ \beta_{-2}^{(j+1)} \end{bmatrix} = \frac{1}{\mu_j(1+\kappa_{j+1})} \times \begin{bmatrix} 2\mu_j+(\kappa_j-1)\mu_{j+1} & (\mu_{j+1}-\mu_j)/R_j^2 \\ 2R_j^2[\mu_j(1-\kappa_{j+1})-\mu_{j+1}(1-\kappa_j)] & 2(\mu_{j+1}-\mu_j)+\mu_j(1+\kappa_{j+1}) \end{bmatrix} \begin{bmatrix} \alpha_0^{(j)} \\ \beta_{-2}^{(j)} \end{bmatrix},$$

$$\begin{bmatrix} \alpha_1^{(j+1)} \\ \alpha_{-1}^{(j+1)} \\ \beta_{-1}^{(j+1)} \\ \beta_{-3}^{(j+1)} \end{bmatrix} = \begin{bmatrix} E_{11}^{(j+1)} & E_{12}^{(j+1)} & E_{13}^{(j+1)} & E_{14}^{(j+1)} \\ E_{21}^{(j+1)} & E_{22}^{(j+1)} & E_{23}^{(j+1)} & E_{24}^{(j+1)} \\ E_{31}^{(j+1)} & E_{32}^{(j+1)} & E_{33}^{(j+1)} & E_{34}^{(j+1)} \\ E_{41}^{(j+1)} & E_{42}^{(j+1)} & E_{43}^{(j+1)} & E_{44}^{(j+1)} \end{bmatrix} \begin{bmatrix} \alpha_1^{(j)} \\ \alpha_{-1}^{(j)} \\ \beta_{-1}^{(j)} \\ \beta_{-3}^{(j)} \end{bmatrix},$$

for $n = 2, 3, \ldots,$

$$\begin{bmatrix} \alpha_n^{(j+1)} \\ \alpha_{-n}^{(j+1)} \\ \beta_{n-2}^{(j+1)} \\ \beta_{-n-2}^{(j+1)} \end{bmatrix} = \begin{bmatrix} F_{11}^{(j+1)} & F_{12}^{(j+1)} & F_{13}^{(j+1)} & F_{14}^{(j+1)} \\ F_{21}^{(j+1)} & F_{22}^{(j+1)} & F_{23}^{(j+1)} & F_{24}^{(j+1)} \\ F_{31}^{(j+1)} & F_{32}^{(j+1)} & F_{33}^{(j+1)} & F_{34}^{(j+1)} \\ F_{41}^{(j+1)} & F_{42}^{(j+1)} & F_{43}^{(j+1)} & F_{44}^{(j+1)} \end{bmatrix} \begin{bmatrix} \alpha_n^{(j)} \\ \alpha_{-n}^{(j)} \\ \beta_{n-2}^{(j)} \\ \beta_{-n-2}^{(j)} \end{bmatrix}, \tag{28}$$

where $E_{ij}^{(j+1)}$ and $F_{ij}^{(j+1)}$ are real constants, generally depending on μ_j, μ_{j+1}, κ_j, κ_{j+1} and R_j. Their explicit expressions are given in the Appendix.

In establishing the first equation of eqns (28), the single-valuedness condition of displacement for the $(j+1)$th layer cylinder, in a form analogous to eqn (23), has been used. In addition, as is discussed in the preceding subsidiary problem, the following relation

$$\left[(\kappa_j-\kappa_{j+1}-1)\frac{\mu_{j+1}}{\mu_j}+1\right]R_j^4\alpha_1^{(j)}+\left\{2\left(1-\frac{\mu_{j+1}}{\mu_j}\right)+\log R_j\left[\frac{\mu_{j+1}}{\mu_j}\kappa_j(1+\kappa_{j+1})-2\kappa_{j+1}\right]\right\}R_j^2\alpha_{-1}^{(j)}$$

$$= \left(1-\frac{\mu_{j+1}}{\mu_j}\right)\beta_{-3}^{(j)}+\log R_j\left[\frac{\mu_{j+1}}{\mu_j}(1+\kappa_{j+1})-2\kappa_{j+1}\right]R_j^2\beta_{-1}^{(j)} \tag{29}$$

must hold to secure the two adjoining layer cylinders in a static equilibrium state. In the special case of $\mu_{j+1} = \mu_j$ and $\kappa_{j+1} = \kappa_j$, relation (29) virtually disappears, since it reduces to the single-valuedness condition of displacement for the jth layer cylinder, which has already been satisfied.

Also, the following equations

$$\alpha_{\pm 1}^{(j)} = \alpha_{\pm 3}^{(j)} = \alpha_{\pm 5}^{(j)} = \cdots = 0, \quad \beta_{\pm 1}^{(j)} = \beta_{\pm 3}^{(j)} = \beta_{\pm 5}^{(j)} = \cdots = 0,$$

$$\alpha_{\pm 1}^{(j+1)} = \alpha_{\pm 3}^{(j+1)} = \alpha_{\pm 5}^{(j+1)} = \cdots = 0, \quad \beta_{\pm 1}^{(j+1)} = \beta_{\pm 3}^{(j+1)} = \beta_{\pm 5}^{(j+1)} = \cdots = 0 \tag{30}$$

hold, when the stress state in the jth layer cylinder, and consequently that in the $(j+1)$th one, are also symmetric to the y-axis.

Now turn to considering the traction boundary problem of the layered hollow circular cylinder, assuming that at the boundary, $r = R_0$, it is stressed by an external load distribution symmetric to the x-axis, while the other boundary, $r = R_N$, is traction free. The case that the boundary, $r = R_N$, is also a loaded one can be treated similarly with superposition.

From eqns (28) it is observed that the coefficients $\alpha_k^{(j)}$ and $\beta_k^{(j)}$ are divided into three groups, $\alpha_0^{(j)}, \beta_{-2}^{(j)}$; $\alpha_{-1}^{(j)}, \beta_{-1}^{(j)}, \beta_{-3}^{(j)}$ and $\alpha_n^{(j)}, \alpha_{-n}^{(j)}, \beta_{n-2}^{(j)}, \beta_{-n-2}^{(j)}$; decoupled from each other. For instance, for $n \geqslant 2$, $\alpha_n^{(j+1)}$, $\alpha_{-n}^{(j+1)}$, $\beta_{n-2}^{(j+1)}$ and $\beta_{-n-2}^{(j+1)}$ are determined by $\alpha_n^{(j)}$, $\alpha_{-n}^{(j)}$, $\beta_{n-2}^{(j)}$ and $\beta_{-n-2}^{(j)}$ alone, and vice versa. Meantime, the coefficients $\alpha_n^{(1)}$, $\alpha_{-n}^{(1)}$, $\beta_{n-2}^{(1)}$ and $\beta_{-n-2}^{(1)}$ relate reciprocally only to A_n, A_{-n}, B_{1+n} and B_{1-n} in eqns (22). On account of this, the problem can be treated individually for the following three cases.

(1) A_0 is given to prescribe the tractions at $r = R_0$, and B_1 should be sought to determine the basic functions $\Phi(z)$ and $\Psi(z)$. Other A_i's and B_i's vanish.

In this case only two kinds of the coefficients, $\alpha_0^{(j)}$ and $\beta_{-2}^{(j)}$, are concerned. Using eqns (21), (28), the following equation comes out straightforwardly

$$\begin{bmatrix} \alpha_0^{(N)} \\ \beta_{-2}^{(N)} \end{bmatrix} = [Q^{(N-1)}]\,[Q^{N-2)}]\cdots[Q^{(1)}]\,[Q^{(0)}] \begin{bmatrix} A_0 \\ B_1 \end{bmatrix}, \tag{31}$$

where $[Q^{(i)}]$, $i = 1, 2, \ldots, N-1$ stands for the 2×2 matrix in the first of eqns (28) with $j = i$, $[Q^{(0)}]$ is a 2×2 matrix, whose elements are

$$[Q_{11}^{(0)}, Q_{12}^{(0)}, Q_{21}^{(0)}, Q_{22}^{(0)}] = (1+\kappa_1)^{-1}[1, 1/R_0, (1-\kappa_1)R_0^2, 2R_0]. \tag{32}$$

On the other hand, since the boundary $r = R_N$ is traction free, we have

$$[2 \quad -1/R_n^2]\,[\alpha_0^{(N)} \quad \beta_{-2}^{(N)}]^T = 0. \tag{33}$$

Therefore, multiplying both sides of eqn (31) with $[2 \quad -1/R_N^2]$ results in a linear algebraic equation to determine the unknown B_1. All coefficients, $\alpha_0^{(i)}$ and $\beta_{-2}^{(i)}$, can subsequently be fixed by using eqns (21), (28).

(2) A_{-1} and A_1 are given, B_0 should be sought. Other A_i's and B_i's vanish.

In this case only $\alpha_1^{(i)}$, $\alpha_{-1}^{(i)}$, $\beta_{-3}^{(i)}$ and $\beta_{-3}^{(i)}$ are involved. However, the coefficient A_{-1} represents a static inequilibrium traction system acting at $r = R_N$; and the only parameter B_0 cannot be adjusted to eliminate the two stress components, σ_r and $\tau_{r\theta}$, at $r = r_N$. As a result, to retain the boundary $r = R_N$ a traction free one both A_{-1} and A_1 must be vanishing.

(3) A_n and A_{-n} ($n = 2, 3, \ldots$) are given, B_{1+n}, B_{1-n} ($n = 2, 3, \ldots$) should be sought. Other A_i's and B_i's vanish.

In this case only $\alpha_n^{(i)}$, $\alpha_{-n}^{(i)}$, $\beta_{n-2}^{(i)}$ and $\beta_{-n-2}^{(i)}$ are involved. To solve the problem we first write, mainly by repeatedly using the last of eqns (28),

$$[\alpha_n^{(N)}, \alpha_{-n}^{(N)}, \beta_{n-2}^{(N)}, \beta_{-n-2}^{(N)}]^T = [L_1]\,[B_{1+n} \quad B_{1-n}]^T + [L_2]\,[A_n \quad A_{-n}]^T, \tag{34}$$

where

$$[L_k] = [F^{(N-1)}]\,[F^{(N-2)}]\cdots[F^{(1)}]\,[H^{(k)}], \quad k = 1, 2, \tag{35}$$

and $[H^{(k)}]$ are 4×2 matrix, whose elements can easily be determined from eqns (22) and are given in the Appendix.

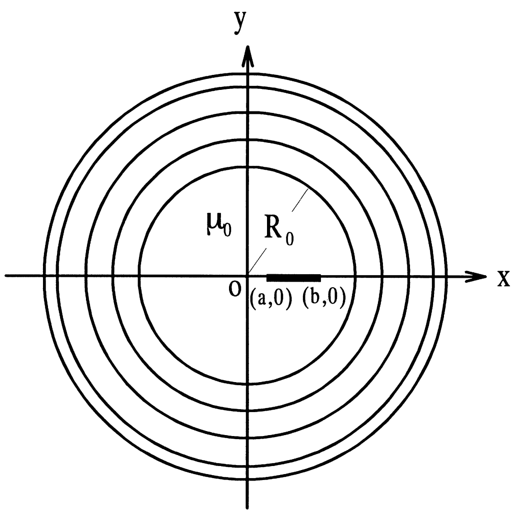

Fig. 4. The composed circular cylinder and the crack.

The traction-free condition on the surface $r = R_N$ gives:

$$[M]\,[\alpha_n^{(N)} \quad \alpha_{-n}^{(N)} \quad \beta_{n-2}^{(N)} \quad \beta_{-n-2}^{(N)}]^T = 0, \tag{36}$$

where $[M]$ is a 2×4 matrix, whose elements can be fixed from the free boundary condition through the first of eqns (20) and are given in the Appendix.

Substituting eqn (34) into (36), with some simple manipulations yields

$$\begin{bmatrix} B_{1+n} \\ B_{1-n} \end{bmatrix} = -([M]\,[L_1])^{-1}([M]\,[L_2]) \begin{bmatrix} A_n \\ A_{-n} \end{bmatrix}. \tag{37}$$

Utilizing the obtained values of B_{1+n} and B_{1-n} in eqns (22), α_n, α_{-n}, β_{n-2} and β_{-n-2} are determined, and this solves the problem.

In this section the displacements and stresses in the adjoining layer cylinders are linked by the continuity conditions in the form of matrix formulas. As indicated in the introduction section, using matrix to link various quantities of adjoining components in a system seems first being employed in the electric circuit theory, later this approach has also been used in structural dynamics and other disciplines. The content of this section is technically new, since it is the first time to combine the complex function method and the matrix presentation to yield the exact solution for a circular cylinder composed of layer cylinders.

4. A crack in a composite circular cylinder

Consider a composite circular cylinder, consisting of the layered hollow circular cylinder, shown in Fig. 3, and a cracked circular cylinder by perfect bonding, as depicted in Fig. 4. We treat the case when the outer boundary of the composite circular cylinder is traction free and the x-axis is a symmetric axis to the problem. All other cases can be treated similarly with superposition.

To solve the problem, suppose that normal traction acting on the crack surfaces is prescribed by

$$f(t) = \sum_{i=0}^{n} f_i t^i, \tag{38}$$

and write $\Phi(z)$ and $\Psi(z)$, in the form of eqn (19), as the two basic stress functions for the cracked circular cylinder, valid in the region $|z| > |a|, |b|$. The latter can be written by using solution (9) and the following formula (Muskhelishvili, 1953)

$$\Psi(z) = \bar{\Omega}(z) - \Phi(z) - z\Phi'(z). \tag{39}$$

As a result, each coefficient α_i or β_i in the two stress functions can be expressed via a linear combination of the unknown constants a_i and c_i in eqn (9), and the given coefficients f_i in eqn (38). In matrix form, this linear combination is written as

$$[R] = [P^{(1)}][G] + [P^{(2)}][Q], \tag{40}$$

where $[R]$, $[G]$ and $[Q]$ are $4n \times 1$ matrices and

$$[R] = [\alpha_0, \beta_{-2}, \alpha_1, \beta_{-3}, \alpha_2, \alpha_{-2}, \beta_0, \beta_{-4}, \alpha_3, \alpha_{-3}, \beta_1, \beta_{-5}, \ldots, \alpha_n, \alpha_{-n}, \beta_{n-2}, \beta_{-n-2}]^T,$$

$$[G] = [a_0, c_0, a_1, c_1, a_2, c_2, \ldots, a_{2n-1}, c_{2n-1}]^T,$$

$$[Q] = \left[f_0, f_1, f_2, \ldots, f_n, \underbrace{0, 0, \ldots, 0}_{3n-1 \text{ entries}} \right]^T, \tag{41}$$

where $P^{(1)}$ and $P^{(2)}$ are $4n \times 4n$ matrices, whose elements can easily be determined by using eqns (8), (9), (38), and are given in the Appendix. The coefficients α_{-1} and β_{-1}, being vanishing ones due to the single-valuedness condition (11), are not included in $[R]$.

Equation (40) gives the coefficients α_i and β_i for the cracked circular cylinder, which will be denoted as $\alpha_i^{(0)}$ and $\beta_i^{(0)}$. By repeatedly using eqns (28), $\alpha_i^{(N)}$ and $\beta_i^{(N)}$, the coefficients for the outer layer cylinder, can be determined from $\alpha_i^{(0)}$ and $\beta_i^{(0)}$ via the following formula

$$[R^{(N)}] = [E^{(N-1)}][E^{(N-2)}] \ldots [E^{(1)}][E^{(0)}][R], \tag{42}$$

where $[R^{(N)}]$ is a $4n \times 1$ matrix, obtained from $[R]$ by replacing α_i and β_i with $\alpha_i^{(N)}$ and $\beta_i^{(N)}$, respectively; $[E^{(i)}]$, $i = 0, 1, \ldots, N-1$, are square matrices of order $4n$. They are in fact overall matrices, assembled from sub-matrices presented in eqns (28). As illustrations, we have

$$E_{11}^{(N-1)} = \frac{2\mu_{N-1} + (\kappa_{N-1} - 1)\mu_N}{\mu_{N-1}(1+\kappa_N)}, \quad E_{12}^{(N-1)} = \frac{\mu_N - \mu_{N-1}}{R_{N-1}^2 \mu_{N-1}(1+\kappa_N)}, \quad E_{55}^{(N-1)} = F_{11}^{(n)},$$

$$E_{11}^{(0)} = \frac{2\mu_0 + (\kappa_0 - 1)\mu_1}{\mu_0(1+\kappa_1)}, \quad E_{12}^{(0)} = \frac{\mu_1 - \mu_0}{R_0^2 \mu_0(1+\kappa_0)}, \quad E_{55}^{(0)} = F_{11}^{(1)} = \frac{\mu_0 + \mu_1 \kappa_0}{\mu_0(1+\mu_1)} \tag{43}$$

other elements of $[E^{(i)}]$ can be written down similarly.

Table 1
Numerical values of normalized stress intensity factors $K_{\mathrm{I}}/(\sigma_0\sqrt{a})$

a	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
Free boundary	1.0000	1.0150	1.0600	1.1356	1.2431	1.3872	1.5783	1.8403	2.2384	3.0382
Fixed boundary	1.0000	0.9852	0.9437	0.8831	0.8125	0.7396	0.6697	0.6054	0.5460	0.4842

Since the outer boundary is free, we have

$$[S]\,[R^{(N)}] = 0 \tag{44}$$

where $[S]$ is a $4n \times 4n$ matrix, whose elements can be found by using the free boundary condition and the first of eqns (20), and are presented in the Appendix. Note the single-valuedness of the displacement has automatically been satisfied in this formulation.

Finally, substituting eqns (40), (42) into (44) and carrying out some simple manipulations, the following equation is obtained to fix the basic unknown matrix $[G]$

$$[G] = [T_1]\,[T_2]\,[Q], \tag{45}$$

where

$$\begin{aligned} [T_1] &= \{[S]\,[E^{(N-1)}]\,[E^{(N-2)}]\ldots[E^{(0)}]\,[P^{(1)}]\}^{-1}, \\ [T_2] &= -[S]\,[E^{(N-1)}]\,[E^{(N-2)}]\ldots[E^{(0)}]\,[P^{(2)}]. \end{aligned} \tag{46}$$

Once $[G]$ is determined, displacements and stresses in the cracked circular cylinder can be found from eqns (9), (10), (14), and those in the layer cylinders are fixed successively by applying eqns (19)–(22), (28).

5. Computation practice and discussion

In this section a number of numerical results, worked out from the solutions developed in the preceding sections, are presented to corroborate that the developed solutions are valid in analysis, and amenable to numerical treatment in computation practice to yield knowledge and information of engineering interest.

The stress intensity factors for a circular cylinder of unit radius having a central crack of length $2a$, and with a free or fixed outer boundary, are given in Table 1, when the crack surfaces are opened by a uniform compressive traction of magnitude σ_0. These values represent the upper (free boundary) and lower (fixed boundary) bounds of the stress intensity factor of a cracked circular cylinder with an elastically constrained outer boundary. The obtained numerical values of stress intensity factors for the free outer boundary can be compared with those recorded in literature (Shangchow, 1983), with the result of a complete agreement.

The solution developed in Section 3 for a layered hollow circular cylinder is of basic significance in the mechanics of composite materials. The numerical results associated with the solution belong

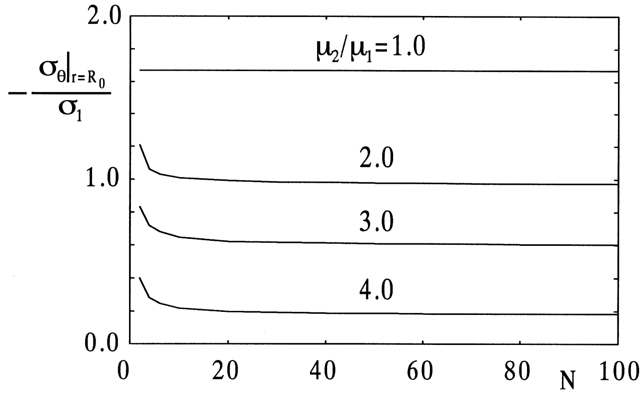

Fig. 5. Numerical values of normalized circumferential stress $-\sigma_\theta|_{r=R_0}/\sigma_1\,(\sigma_r|_{r=R_0}=\sigma_1)$.

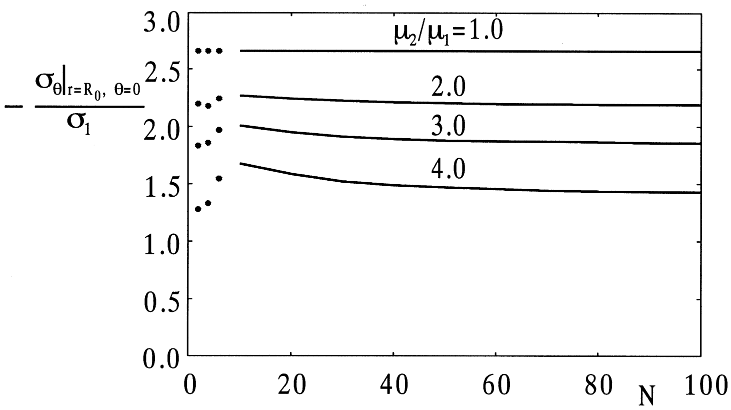

Fig. 6. Numerical values of normalized circumferential stress $\sigma_\theta|_{r=R_0,\theta=0}/\sigma_1\,(\sigma_r|_{r=R_0}=2\sigma_1\cos 4\theta)$.

to a layered hollow circular cylinder, of inner radius $R_0 = 1$ and outer radius $R_N = 2$, consisting of $N = 2, 4, \ldots, 100$ layer hollow circular cylinders of uniform thickness, with $\mu_1 = \mu_3 = \cdots = \mu_{N-1}$, $\mu_2 = \mu_4 = \cdots = \mu_N$, $v_1 = v_2 = \cdots = v_N = 1/3$. The following traction boundary conditions

$$r = R_0, \quad \sigma_r = \sigma_1, \quad \text{or } 2\sigma_1\cos 4\theta, \quad \tau_{r\theta} = 0; r = R_N, \quad \sigma_r = \tau_{r\theta} = 0 \tag{47}$$

are considered, and the value of $\sigma_\theta|_{r=R_0,\theta=0}$ is selected to represent the stress distribution within the layered cylinder as shown in Figs 5 and 6.

It is seen from the figures that in both cases ($\sigma_r = \sigma_1$ and $2\sigma_1\cos 4\theta$), the values of σ_θ approach a stationary value for large N. For $\sigma_r = \sigma_1$, which is a simple mode of traction distribution, the approaching is monotonic. While for $\sigma_r = 2\sigma_1\cos 4\theta$, the approaching is relatively slow, and has a turn at small N shown by points. This fact indicates that the stress distribution in the layered cylinder definitely approach a stationary one as the number N increases. It is this stationary stress

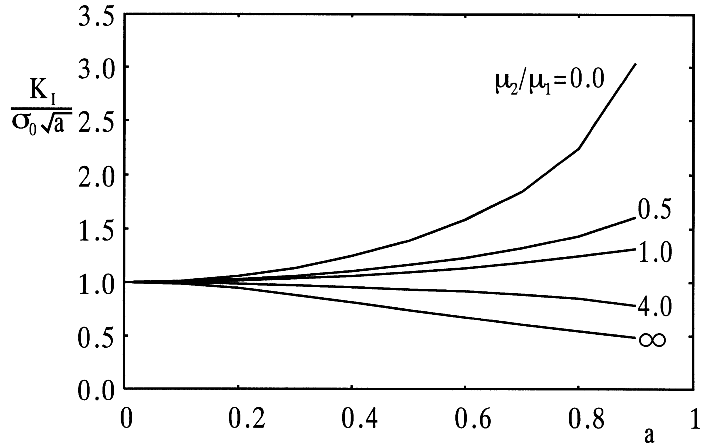

Fig. 7. Numerical values of normalized stress intensity factors.

Table 2
Numerical values of normalized stress intensity factors $K_I/(\sigma_0\sqrt{a})$

a		0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
N	4	1.0000	1.0008	1.0034	1.0086	1.0165	1.0275	1.0407	1.0541	1.0627	1.0489
	8	1.0000	1.0010	1.0044	1.0117	1.0200	1.0325	1.0476	1.0633	1.0753	1.0667
	12	1.0000	1.0011	1.0048	1.0114	1.0212	1.0343	1.0500	1.0666	1.0800	1.0742
	16	1.0000	1.0012	1.0050	1.0118	1.0218	1.0351	1.0512	1.0683	1.0826	1.0792
	20	1.0000	1.0012	1.0051	1.0120	1.0222	1.0357	1.0519	1.0694	1.0842	1.0822
	40	1.0000	1.0013	1.0053	1.0124	1.0229	1.0367	1.0534	1.0715	1.0874	1.0886
	60	1.0000	1.0013	1.0054	1.0126	1.0231	1.0371	1.0540	1.0722	1.0885	1.0908
	80	1.0000	1.0013	1.0054	1.0126	1.0233	1.0373	1.0542	1.0726	1.0891	1.0919
	100	1.0000	1.0013	1.0054	1.0127	1.0233	1.0374	1.0544	1.0728	1.0894	1.0926

distribution that can be used to accurately describe the deformation behavior of a periodically stacked layered structure.

The cracked circular cylinder, of shear modulus μ_0, and the layered hollow circular cylinder, both having been considered above, are now perfectly bonded to each other to form a composed circular cylinder shown in Fig. 4. Numerical values of stress intensity factor are first evaluated for the special case $\mu_1 = \mu_2$, that is, when the layered hollow circular cylinder is a homogeneous one. The result is shown in Fig. 7. Computations are then made on another case, where $\mu_1/\mu_0 = 2.0$ and $\mu_2 = \mu_0$ with different values of N, the layer number. The result is shown in Table 2.

It is seen from Fig. 7 that values for $\mu_1/\mu_0 = 0$ (free boundary) and ∞ (fixed boundary) stand for the upper and lower bounds of the stress intensity factors, respectively. In Table 2, the phenomenon that when N becomes large, K_I approaches a stationary value is once again observed.

These numerical data, highly accurately computed for a fully heterogeneous elastic object, manifests the effectiveness and usefulness of an analytic method.

Our analytical development as well as numerical practice has been carried out on the basis that the problem is symmetric to the x-axis. However, with some obvious algebraic manipulations it is easy to do to extend the result into the case, when the problem is antisymmetric to the x-axis. The solution to the general case then can be obtained by the superposition of these two types of solutions. On the other hand, while this investigation considers a composed cylinder made of different and isotropic materials, the basic procedure developed in the solution can also be used to deal with composed cylinders consisting of different and anisotropic materials, especially for materials that are orthotropic with respect to the axial, radial and circumferential directions.

Appendix

(A) Expressions for the real coefficients B_i in eqn (8)

$$B_0 = 1, \quad B_n = \sum_{k=0}^{n} a_k b_{n-k}, \quad n = 2, 3, \ldots,$$

$$a_0 = b_0 = 1, \quad (a_n, b_n) = \frac{(2n-1)!!}{(2n)!!}(a^n, b^n), \quad n = 1, 2, \ldots. \tag{A1}$$

(B) Expressions for the elements $E_{mn}^{(j+1)}$ in eqn (28)

$$E_{11}^{(j+1)} = \frac{\mu_{j+1}}{\mu_j}, \quad E_{12}^{(j+1)} = \frac{\log R_j}{R_j^2}\left(\frac{2\kappa_{j+1}}{1+\kappa_{j+1}} - \frac{\mu_{j+1}\kappa_j}{\mu_j}\right),$$

$$E_{13}^{(j+1)} = \frac{\log R_j}{R_j^2}\left(\frac{\mu_{j+1}}{\mu_j} - \frac{2\kappa_{j+1}}{1+\kappa_{j+1}}\right), \quad E_{14}^{(j+1)} = 0,$$

$$E_{21}^{(j+1)} = E_{24}^{(j+1)} = 0, \quad E_{22}^{(j+1)} = -E_{23}^{(j+1)} = \frac{1}{1+\kappa_{j+1}},$$

$$E_{31}^{(j+1)} = E_{34}^{(j+1)} = 0, \quad E_{32}^{(j+1)} = -E_{33}^{(j+1)} = -\frac{\kappa_{j+1}}{1+\kappa_{j+1}},$$

$$E_{41}^{(j+1)} = R_j^4\left(\frac{\mu_{j+1}}{\mu_j} - 1\right), \quad E_{42}^{(j+1)} = R_j^2\left[\log R_j\left(\frac{2\kappa_{j+1}}{1+\kappa_{j+1}} - \frac{\mu_{j+1}\kappa_j}{\mu_j}\right) - \frac{2\kappa_{j+1}}{1+\kappa_{j+1}}\right],$$

$$E_{43}^{(j+1)} = R_j^2\left[\log R_j\left(\frac{\mu_{j+1}}{\mu_j} - \frac{2\kappa_{j+1}}{1+\kappa_{j+1}}\right) - \frac{2}{1+\kappa_{j+1}}\right], \quad E_{44}^{(j+1)} = 1. \tag{A2}$$

(C) Expressions for the elements $F_{mn}^{(j+1)}$ in eqn (28)

$$F_{11}^{(j+1)} = \frac{\mu_j + \mu_{j+1}\kappa_j}{\mu_j(1+\kappa_{j+1})}, \quad F_{12}^{(j+1)} = \frac{1+n}{R_j^{2n}} \cdot \frac{\mu_j - \mu_{j+1}}{\mu_j(1+\kappa_{j+1})},$$

$$F_{13}^{(j+1)} = 0, \quad F_{14}^{(j+1)} = \frac{\mu_{j+1}-\mu_j}{\mu_j(1+\kappa_{j+1})R_j^{2n+2}},$$

$$F_{21}^{(j+1)} = \frac{(1-n)R_j^{2n}}{1+\kappa_{j+1}}\left(1-\frac{\mu_{j+1}}{\mu_j}\right), \quad F_{22}^{(j+1)} = F_{11}^{(j+1)},$$

$$F_{23}^{(j+1)} = \frac{R_j^{2n-2}}{1+\kappa_{j+1}}\left(\frac{\mu_{j+1}}{\mu_j}-1\right), \quad F_{24}^{(j+1)} = 0,$$

$$F_{31}^{(j+1)} = \frac{(1-n)R_j^2}{1+\kappa_{j+1}}\left[1-\kappa_{j+1}-\frac{\mu_{j+1}}{\mu_j}(1-\kappa_j)\right],$$

$$F_{32}^{(j+1)} = \frac{R_j^{2-2n}}{1+\kappa_{j+1}}\left[(n^2-1)\left(\frac{\mu_{j+1}}{\mu_j}-1\right)+\frac{\kappa_j\mu_{j+1}}{\mu_j}-\kappa_{j+1}\right],$$

$$F_{33}^{(j+1)} = \frac{\mu_{j+1}+\mu_j\kappa_{j+1}}{\mu_j(1+\kappa_{j+1})}, \quad F_{34}^{(j+1)} = \frac{1-n}{(1+\kappa_{j+1})R_j^{2n}}\left(\frac{\mu_{j+1}}{\mu_j}-1\right),$$

$$F_{41}^{(j+1)} = \frac{R_j^{2n+2}}{1+\kappa_{j+1}}\left[(1-n^2)\left(1-\frac{\mu_{j+1}}{\mu_j}\right)-\kappa_{j+1}+\frac{\kappa_j\mu_{j+1}}{\mu_j}\right],$$

$$F_{42}^{(j+1)} = \frac{(1+n)R_j^2}{1+\kappa_{j+1}}\left[1-\kappa_{j+1}+\frac{\mu_{j+1}}{\mu_j}(\kappa_j-1)\right],$$

$$F_{43}^{(j+1)} = \frac{(1+n)R_j^{2n}}{1+\kappa_{j+1}}\left(\frac{\mu_{j+1}}{\mu_j}-1\right), \quad F_{44}^{(j+1)} = F_{33}^{(j+1)}. \tag{A3}$$

(D) Expressions for the elements $H_{ij}^{(k)}$ in eqn (35)

$$H_{11}^{(1)} = \frac{1+n}{(1+\kappa_1)R_0^{n+1}}, \quad H_{12}^{(1)} = H_{21}^{(1)} = 0, \quad H_{22}^{(1)} = \frac{(1-n)R_0^{n-1}}{1+\kappa_1},$$

$$H_{31}^{(1)} = \frac{1-n^2}{(1+\kappa_1)R_0^{n-1}}, \quad H_{32}^{(1)} = \frac{(1-n)}{(1+\kappa_1)R_0^{n-1}},$$

$$H_{41}^{(1)} = \frac{(1+n)R_0^{n+1}}{1+\kappa_1}, \quad H_{42}^{(1)} = \frac{(1-n^2)R_0^{n+1}}{1+\kappa_1},$$

$$H_{11}^{(2)} = H_{22}^{(2)} = 0, \quad H_{12}^{(2)} = \frac{1}{(1+\kappa_1)R_0^n}, \quad H_{21}^{(2)} = \frac{R_0^n}{1+\kappa_1},$$

$$H_{31}^{(2)} = -\frac{\kappa_1}{(1+\kappa_1)R_0^{n-2}}, \quad H_{32}^{(2)} = \frac{1-n}{(1+\kappa_1)R_0^{n-2}},$$

$$H_{41}^{(2)} = \frac{(1+n)R_0^{n+2}}{(1+\kappa_1)}, \quad H_{42}^{(2)} = -\frac{\kappa_1 R_0^{n+2}}{(1+\kappa_1)}. \tag{A4}$$

(E) Expressions for the elements M_{ij} in eqn (36)

$$M_{11} = (1-n)R_N^n, \quad M_{12} = R_N^{-n}, \quad M_{13} = -R_N^{n-2}, \quad M_{14} = 0,$$
$$M_{21} = R_N^n, \quad M_{22} = (1+n)R_N^{-n}, \quad M_{23} = 0, \quad M_{24} = -R_N^{-(n+2)}. \tag{A5}$$

(F) Expressions for the elements $P_{ij}^{(1)}$ in eqn (40)

$$P_{1,1}^{(1)} = 1/2, \quad P_{1,2i}^{(1)} = B_{i-2}, \quad i = 2, 3, \ldots, 2n;$$
$$P_{2,2i}^{(1)} = 2B_i, \quad i = 1, 2, \ldots, 2n;$$
$$P_{3,3}^{(1)} = 1/2, \quad P_{3,2i}^{(1)} = B_{i-3}, \quad i = 3, 4, \ldots, 2n;$$
$$P_{4,2i}^{(1)} = 3B_{i+1}, \quad i = 1, 2, \ldots, 2n;$$
$$\text{for } j = 1, 2, \ldots, n-1,$$
$$P_{4j+1,2j+3}^{(1)} = 1/2, \quad P_{4j+1,2(i+j+3)}^{(1)} = B_i, \quad i = 0, 1, \ldots, 2n-j-3;$$
$$P_{4j+2,2i}^{(1)} = B_{i+j-1}, \quad i = 1, 2, \ldots, 2n;$$
$$P_{4j+3,2j-1}^{(1)} = -(1+j)/2, \quad P_{4j+3,2(i+j+3)}^{(1)} = (1-j)B_i, \quad i = 0, 1, \ldots, 2n-j-3;$$
$$P_{4j+4,2i}^{(1)} = (3+j)B_{i+j-1}, \quad i = 1, 2, \ldots, 2n. \tag{A6}$$

All other elements that do not appear in the above expressions are zero elements.

(G) Expressions for the elements $P_{ij}^{(2)}$ in eqn (40)

$$P_{1,1}^{(2)} = 1/2, \quad P_{3,3}^{(2)} = 1/2;$$
$$\text{for } j = 1, 2, \ldots, n-1, P_{4j+1,j+2}^{(2)} = 1/2;$$
$$\text{for } j = 2, 3, \ldots, n-1, P_{4j+3,j}^{(2)} = (1-j)/2. \tag{A7}$$

All other elements that do not appear in the above expressions are zero elements.

(H) Expressions for the elements S_{ij} in eqn (44)

$$S_{1,1} = 2, \quad S_{1,2} = -R_N^{-2}, \quad S_{2,3} = R_N, \quad S_{2,4} = -R_N^{-3};$$
$$\text{for } i = 2, 3, \ldots, 2n,$$
$$S_{2i-1,4i-3} = (1-i)R_N^i, \quad S_{2i-1,4i-2} = R_N^{-i}, \quad S_{2i-1,4i-1} = -R_N^{i-2};$$
$$S_{2i,4i-3} = R_N^i, \quad S_{2i,4i-2} = (1+i)R_N^{-i}, \quad S_{2i,4i} = -R_N^{-i-2}. \tag{A8}$$

All other elements that do not appear in the above expressions are zero elements.

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